

The Zero Power

Why is $x^0 = 1$ for any number x ?

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First, let's look at a number of improper fractions:

Fraction	Value
$\frac{4}{2}$	2
$\frac{8}{2}$	4
$\frac{16}{2}$	8
$\frac{16}{4}$	4
$\frac{16}{8}$	2

I use numbers that are all powers of 2 because I want to draw attention to *just* the difference of exponents, and not muddle things up with other factors

Next, let's write the fractions and values as powers of 2:

Fraction	Fraction Powers	Value Powers	Value
$\frac{4}{2}$	$\frac{2^2}{2^1}$	2^1	2
$\frac{8}{2}$	$\frac{2^3}{2^1}$	2^2	4
$\frac{16}{2}$	$\frac{2^4}{2^1}$	2^3	8
$\frac{16}{4}$	$\frac{2^4}{2^2}$	2^2	4
$\frac{16}{8}$	$\frac{2^4}{2^3}$	2^1	2

Okay, there may be a pattern here. Consider the $\frac{16}{2} = 8$, is there an equation that shows a pattern among the exponents:

$$\frac{2^{\boxed{4}}}{2^{\boxed{1}}} = 2^{\boxed{3}} \implies 4^{\boxed{?}} = 3$$

What operation (from $\times \div + -$) can fill in the $\boxed{?}$ to make the above equation true? Will this work for all of the other fractions in the table?

Fraction	Fraction Powers	Pattern?	Value Powers	Value
$\frac{4}{2}$	$\frac{2^2}{2^1}$	$2 - 1 = 1$	2^1	2
$\frac{8}{2}$	$\frac{2^3}{2^1}$	$3 - 1 = 2$	2^2	4
$\frac{16}{2}$	$\frac{2^4}{2^1}$	$4 - 1 = 3$	2^3	8
$\frac{16}{4}$	$\frac{2^4}{2^2}$	$4 - 2 = 2$	2^2	4
$\frac{16}{8}$	$\frac{2^4}{2^3}$	$4 - 3 = 1$	2^1	2

Subtraction seems to work! But can you explain why it works?

Let's simplify the following three fractions:

$$\frac{60}{20}$$

$$\frac{20}{35}$$

$$\frac{8}{32}$$

Now, let's simplify them a different way, by using the *prime factorization* of each number:

$$\frac{60}{20} = \frac{2 \times 2 \times 3 \times 5}{2 \times 2 \times 5} = \frac{2^2 \times 3^1 \times 5^1}{2^2 \times 5^1} \Rightarrow$$

Factor	Power Difference	Power
2	2 - 2	0
3	1 - 0	1
5	1 - 1	0

$$\Rightarrow 3^1 = 3$$

$$\frac{20}{35} = \frac{2 \times 2 \times 5}{5 \times 7} = \frac{2^2 \times 5^1}{5^1 \times 7^1} \Rightarrow$$

Factor	Power Difference	Power
2	2 - 0	2
5	1 - 1	0
7	0 - 1	-1

$$\Rightarrow 2^2 \times 7^{-1} = \frac{2^2}{7^1} = \frac{4}{7}$$

$$\frac{8}{32} = \frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{2^3}{2^5} \Rightarrow$$

Factor	Power Difference	Power
2	3 - 5	-2

$$\Rightarrow 2^{-2} = \frac{1}{4}$$

You've probably done this type of fraction simplification before; the beauty of this technique is that using prime factorization is the *only* way to be *sure* that some big ugly fraction (maybe $\frac{1870}{5187}$) is actually in simplest terms. But we can also use this to show that x^0 must be 1! Let's look again at our calculations for the prime factorization simplifications, we could have also written the first two like this:

this factors into $\frac{2 \times 5 \times 11 \times 17}{3 \times 7 \times 13 \times 19}$

$$\frac{60}{20} = \frac{2 \times 2 \times 3 \times 5}{2 \times 2 \times 5} = 2^0 \times 3^1 \times 5^0 = 3$$

$$\frac{20}{35} = \frac{2 \times 2 \times 5}{5 \times 7} = 2^2 \times 5^0 \times 7^{-1} = \frac{4}{7}$$

From these, it is clear that 2^0 and 5^0 must be 1, *otherwise the math would not work*. If 2^0 and 5^0 did not equal 1, then $\frac{60}{20}$ would not equal 3 and $\frac{20}{35}$ would not equal $\frac{4}{7}$.

Finally, let's consider one more example, this time a more abstract representation. Suppose you have a fraction

$$\frac{x^m}{x^n} = \frac{\overbrace{x \times x \times x \times \dots \times x}^{m \text{ times}}}{\underbrace{x \times x \times x \times \dots \times x}_{n \text{ times}}} \Rightarrow$$

Factor	Power Difference	Power
x	$m - n$	$m - n$

$$\Rightarrow x^{m-n}$$

There are three cases we must consider:

1. If $m > n$ then there are more factors of x in the numerator and the result is similar to $\frac{60}{20}$ above; it is a whole number.

2. If $m < n$ then there are more factors of x in the denominator and the result is similar to the $\frac{8}{32}$ above; it is a fraction between 0 and 1.
3. If $m = n$ then we have the special case of x^0 . But if $m = n$ then we have a fraction with the same number in the numerator as the denominator, and we already know the value is 1, so $x^0 = 1$.